**PNAS Template for Main Manuscript**

This PNAS template for the Main Manuscript may be used to organize your main text source file. The template is intended to provide a clearly organized PDF to facilitate the review process. Further information is available in our [Author Center](https://www.pnas.org/authors/submitting-your-manuscript#article-types).

**Using the template**

Paste the appropriate text from your manuscript into the relevant section of the template. You may maintain the template formatting, or reapply styles after pasting your text into the template.

Figures should be placed on separate pages with legends set immediately below each figure. Table titles should be set immediately above each table. Please do not use field codes for figures and tables to ensure the numbering is correct after the PDF conversion.

References cited in the main text should be included in a separate reference list at the end of this file. Examples of the PNAS citation style are included below.

**Notes about submission**

The following items are required on the title page: **Title, Author Line, Author Affiliations, Corresponding Author information**. Each of these items **must** be provided on the title page for us to proceed with processing your paper.

You are not required to adhere to the section order outlined below. For example, you may combine your Results and Discussion or use alternate section headings. Materials and Methods should be included after the Results and Discussion in most cases. If your paper does not include the standard section headings, please provide a brief explanation in the **Comments to Editorial Staff** field of the submission form.

You may include subheadings within the standard headings listed below. You may also include line numbers.

**Submitting your main manuscript**

Delete this first page, and then save your completed main text file as a PDF for submission, following instructions available [here](http://www.pnas.org/site/authors/procedures.xhtml#preparation).

*Updated February 2022*



**Main Manuscript for**

Behavioral and Topological Heterogeneities in Network Versions of Schelling’s Segregation Model

Will Deter, Hiroki Sayama

Binghamton Center of Complex Systems, Department of Systems Science and Industrial Engineering, Binghamton University, State University of New York

\*Paste corresponding author name(s) here.

**Email:**  [wdeter1@binghamton.edu](mailto:wdeter1@binghamton.edu), sayama@binghamton.edu

PNAS strongly encourages authors to supply an [ORCID identifier](https://orcid.org/) for each author. Do not include ORCIDs in the manuscript file; individual authors must link their ORCID account to their PNAS account at [www.pnascentral.org](http://www.pnascentral.org). For proper authentication, authors must provide their ORCID at submission and are not permitted to add ORCIDs on proofs.

**Author Contributions:** Paste the author contributions here.

**Competing Interest Statement:** Disclose any competing interests here.

**Classification:** Paste the major and minor classification here. Dual classifications are permitted, but cannot be within the same major classification.

**Keywords:** segregation models, agent heterogeneity, topological heterogeneity

**This PDF file includes:**

Main Text

Figures 1 to X

Tables 1 to X

**Abstract**

Agent-based network models of residential segregation have been of persistent interest to various research communities since their origin with Thomas Schelling. Frequently, these models have sought to elucidate the extent to which the collective dynamics of individual preferences may cause segregation to emerge. This open question has sustained relevance in U.S. jurisprudence. Previous investigation of heterogeneity of behaviors (preferences) by Xie & Zhou (2012) has shown reductions in segregation on networks. Previous investigation of heterogeneity of topologies by Gandica, Gargiulo, & Carletti (2016) has shown no significant impact to observed segregation levels. Recent work by Sayama and Yamanoi (2020) has shown the importance of representing realistic heterogeneities in dynamical social network models. In this work, the necessity of concurrent representation of both behavioral and topological heterogeneities in network segregation models is examined. Extending the previous works, additional network simulations were conducted using both Xie & Zhou’s and Schelling’s preference models on 2D lattices with varied levels of densification to create topological heterogeneities (i.e., clusters, hubs). Results show a richer variety of outcomes, including novel differences in resultant segregation levels, fragmentation, and hub composition. Notably, with concurrent increased representations of heterogeneous preferences and heterogenous topologies, reduced levels of segregation and fragmentation emerge. Implications and areas for future study are discussed.

.

Gandica, Y., Gargiulo, F., & Carletti, T. (2016). Chaos, Solitons, and Fractals, 90, 46-54.

Sayama, H., & Yamanoi, J. (2020). NetSci-X 2020 Proceedings, pp. 171-181.

Xie, Y., & Zhou, X. (2012). PNAS, 109(29), 11646-11651.

**Significance Statement**

This work further develops understanding of micro-level dynamics which collectively produce segregation on networks. It demonstrates the value of incorporating both behavioral and topological heterogeneities in network models of residential segregation. It shows that inclusion of both results in significantly lower levels of resultant segregation. Additionally, this work reveals an additional dynamic: segregation of tolerance levels which is enabled by the heterogeneous topology – the urban-rural divide. These issues are of substantial socio-political interest.

**Main Text**

1. **Introduction**

Residential segregation is a persistent topic of discussion in the social sciences whose investigation is frequently enriched by computational models, particularly agent-based models (ABMs) with network structure [1-11]. Residential segregation appears to be a robust and resilient phenomenon with many possible contributing factors at multiple scales [3]. This work focuses primarily on one of the two traditions of segregation theory identified by Fossett [5]: “individual preferences,” which asserts that segregation is an emergent property arising from the collective dynamics of individuals and their choices in mostly-free housing markets. Identifying and understanding factors and behaviors which contribute to residential segregation has substantial sociopolitical significance. In the decades following Brown v. Board of Education, a legal debate surrounding de jure vs. de facto segregation with proponents of the de facto perspective citing the role of such collective dynamics of individual preferences as the genesis of de facto segregation [10]. Given the long-standing view that government should restrict interventions to cases where de jure segregation is evident, it is clearly important to understand the extent to which de facto segregation might (or might not) emerge as the result of the collective dynamics of individual preferences.

Recent works by Sayama & Yamanoi [7] have highlighted the importance of heterogeneity in network models of social problems, showing clearly that certain emergent, system-level phenomena may only arise with sufficient heterogeneity of agent behaviors and/or characteristics. A review of the literature identified many investigations of component heterogeneity in network models of segregation. Notably, Xie and Zhou [7] and Gandica, Gargiulo, & Carletti [7] examine heterogeneous preferences and heterogeneous topology, respectively. No previous studies of combined dimensions of heterogeneity were identified. Recognizing the potential that multiple dimensions of heterogeneity may be required to observe additional phenomena, this work attempts to examine the impact of the combination of behavioral and topological heterogeneities in network models of residential segregation.

We explore the existing literature related to computational models of residential segregation. Particular attention is given to agent-based models represented on grids, checkerboards, lattices, or other topologies characterizable as networks. Important discoveries of network dynamics related to behavioral and topological heterogeneity are highlighted. We discuss the concept of heterogeneity, its relationship with variety, and its implications for dynamical network models of interaction. A baseline model is constructed following Xie & Zhou’s [1] baseline. The baseline model is then elaborated in two ways: first to introduce heterogeneity of preferences and second to introduce heterogeneity of topology. The results of these implementations are illustrated, observables are characterized, and simulations are enumerated. Finally, results, novel behaviors, and opportunities for future work are discussed.

1. **Background**
   1. **Heterogeneity**

At least since Ashby’s [12] exposition of requisite variety, the importance of variety in systems thinking and modeling has been evident. Then, with Ashby in 1970, Conant [13] showed that an effective regulator of a system must be a model of that system. It is not a stretch to see that the variety of a system’s constituent components or behaviors might influence the variety of its emergent, system-level properties. Hartley information [14] measured in bits, given as:

where N is the number of microstates (cardinality) in , provides a measure of variety. As the cardinality of a set, , increases, the set of all possible subsets of its elements, increases. This power set, , has cardinality . Thus, when interactivity of components is considered, variety increases dramatically:

We distinguish heterogeneity from variety via Shannon entropy [15] which provides a measure of heterogeneity. Shannon entropy occupies a central role in information theory and provides a measure of true uncertainty or randomness. The Shannon entropy of a discrete probability distribution measures the average amount of information needed to represent an event drawn from the distribution. In other words, it characterizes the average information content of the distribution. Unlike Harley’s measure, Shannon’s does not presume that all microstates are equally likely. The Shannon entropy, , of a discrete random variable, , with probability distribution is given by:

This construction reflects the notion that the actualization of variety is an important factor; thus, we define heterogeneity as realized variety. If a particular microstate is possible but so improbable that it is rarely realized, one may say that a set possesses less heterogeneity than variety. An increase in cardinality indicates an increase in variety, however, Shannon entropy makes clear that this increase may or may not result in a complementary increase in heterogeneity. Shannon’s measure avoids this weakness and appropriately reflects that the addition of a microstate that occurs with an infinitesimal frequency does not contribute as much heterogeneity as a state which occurs with a likelihood similar to the average likelihood. In addition, it is important to note that Shannon’s measure also appropriately reflects that while the addition of a microstate that occurs with an outsized frequency increases variety, it also reduces entropy, uncertainty, and heterogeneity.

* 1. **Computational Models of Residential Segregation**
     1. **Sakoda and Schelling**

In 1971, Sakoda [8] and then Schelling [9], proposed lattice-based models of social interaction and, more specifically, segregation. These models can be considered some of the earliest instances of agent-based models (ABMs). While Sakoda’s work faded into relative obscurity, Schelling’s gained great renown [18]. Importantly, these works demonstrated the emergence of macro-level (population-level) phenomena that resulted from micro-level (individual) interactions with neighbors. Both showed that segregation was one of the macro-level properties which could be produced by such emergence.

Schelling’s investigation of segregation was more extensive. It included a detailed discussion of the dynamics of his “spatial proximity” model. Like Sakoda, Schelling’s model randomly arranged individuals and vacancies on a 2D lattice. Each individual was assigned a tolerance threshold of 0.50, that is individuals are unhappy the neighborhood proportion of unlike neighbors exceeds 50 percent. Then, without particular order, individuals are selected for transfer to a vacant location where their tolerance threshold is exceeded. This procedure is repeated until all individuals are satisfied, or until no additional viable moves are available. The results were striking. Even without overt segregationist preferences, the collective dynamics of individuals’ preferences to avoid minority status resulted in near-total segregation.

Additionally, Schelling recognized that varying tolerances (preference schedules) and increasing neighborhood sizes, could generate a greater variety of results, including reduced segregation, in his “bounded-neighborhood” model. This model was not, however, concerned with the properties of the configurations of individuals within the neighborhood. Rather, it only considered whether they chose to remain within or exit.

* + 1. **After Schelling**

In the years after Schelling, substantial replication, extension, and discussion of the model took place [4]. Despite Schelling’s acknowledgment of his model’s crudity, it gained importance in the 1980s and 1990s with debates concerning the genesis and perpetuation of segregation. On one side were proponents of the notion that segregation in the United States was primarily caused and perpetuated by discriminatory housing policies. The other side suggested that could be caused by the collective dynamics of ethnic preferences aided by economic disparities [4]. The suggestion that segregation exists due to such collective dynamics has even been an important factor in U.S. court decisions as legal remedies to racial injustices often require evidence of de jure causes [19]. As this controversy unfolded, several researchers examined and elaborated Schelling’s work.

* + - 1. **Empirically Varied Tolerance Schedules**

Among them, Clark [3] built upon Schelling’s “bounded-neighborhood” work by establishing tolerance schedules based on empirical data. Clark was able to provide validation for Schelling’s suggestion that heterogeneous tolerances could lead to a greater variety of results. However, Clark also predicted that mixed equilibria may be rarer than previously expected. It is important to note that this hypothesis applies to a single neighborhood without regard to its specific spatial configuration.

Later, Xie & Zhou [1] adopted the Detroit data used by Bruch & Mare [2] taken from the Multi-City Study of Urban Inequality (MCSUI) to assign tolerance schedules to individuals based on a constructed Guttman scale. Xie & Zhou’s baseline model generated six classes of agents, five following the upper tolerance thresholds of each level of the Guttman scale and a sixth following a rank-ordered logit model. Recognizing the unrealistic assumption of homogeneity within each level of the Guttman scale, Xie & Zhou developed a continuous tolerance schedule by drawing from a uniform distribution bounded by each level’s lower and upper tolerance threshold.

Using this tolerance schedule, Xie & Zhou extended Schelling’s [9] and Bruch & Mare’s [2] work by conducting additional simulations. Their simulations employed transition rules in the same way as their predecessors but used the updated tolerance schedule. Their results demonstrated a substantial reduction in realized segregation resulting from the novel tolerance schedule.

* + - 1. **Parameter Elaboration**

Fossett [5] extensively elaborated Schelling’s “spatial proximity” model to include a variety of additional parameter settings which enabled new experiments. Fossett established a random baseline model with a modified 2D lattice structure which produced no significant segregation. This structure left certain corner units of lattice subsections empty resulting in a rounded superstructure. Next, Fossett successively included additional agent parameters, including tolerances for status, ethnicity, and housing quality. Fossett also examined the impact of reducing agents’ perceptual ranges. Overall, Fossett’s work reinforced the view that micro-level interactions could generate and perpetuate segregation even in the absence of discrimination.

* + - 1. **Game-Theoretic Approach**

Zhang [10] translated Schelling’s “spatial proximity” model to a spatial game-theoretic model. In this, Zhang presumed asymmetrical tolerances between groups and added a simple housing market, again employing a 2D lattice. As prices in the housing market respond to demand, the asymmetry of tolerances plays an important role. Housing for the group with more exclusionary tolerances will be scarcer. Additionally, the quantity of housing units that are *unsuitable* for the exclusionary group may begin to exceed demand. The result is a significant disparity in prices based on neighborhood composition. Zhang shows that in the absence of such market influences, asymmetry of tolerances alone cannot explain the genesis of segregation on the lattice.

In a separate paper, Zhang [11] employs his game-theoretic approach on a 2D lattice to illustrate an extreme example: the genesis of segregation when all agents prefer total integration. This time, no vacancies were permitted on the lattice. Again, the asymmetry of tolerances is important. For each group, Zhang establishes a utility function with maximum payoff when there is a perfect mix of neighbors. Secondarily, when a perfect mix of neighbors is not available, agents prefer to avoid minority status. Zhang shows that this asymmetry of tolerances makes perfect mixing an unstable attractor. Whenever an agent finds a trading partner, the payoff gained by the former will always be outweighed by the penalty to the latter.

* + - 1. **Metapopulation Model**

From 2015 to 2018, Gandica, Gargiulo, & Carletti [6-7] elaborated Schelling’s spatial proximity model using a metapopulation framework. Initially [6], they constructed regular 1- and 2-D lattices. Each node on the lattice could house a population of individuals with some limit . Thus, for each individual living in the th node, its neighborhood size, , the level of population in plus the sum of the levels, , of population in each neighbor, . , where signifies the set of nodes in the neighborhood . The authors conducted simulations using several homogeneous tolerance thresholds, . For , at the node level, total segregation was observed.

It is important to note that the proposed metapopulation model is analogous to the instantiation of a heterogeneous topology superimposed on a lattice structure. To clarify, let , the node population limit be 4. Then, the set of possible node population levels, . For a regular 2D lattice with von Neumann neighborhoods, the range is 1 to 20, or . This is equivalent to adding edges to a 1- or 2-D lattice populated by single occupant nodes to achieve the same heterogeneity of neighborhood sizes, except that the method proposed by Gandica, Gargiulo, & Carletti increases the number of individual neighborhoods.

Later [7], the authors elaborated their earlier work on 2D lattices to examine the effect of varying network structures while employing the metapopulation model. While maintaining an average node-neighborhood size, , small-world (Watts-Strogatz [20]), random (Erdos-Renyi [20]), and scale-free (Barabasi-Albert [22]) networks were generated. Given the same node-level population limit, , a greater variety of neighborhood sizes are possible. Results from this inquiry showed no qualitative difference in asymptotic averaged node magnetization, behavior, where

1. **Models**

As with previous investigations, experiments are conducted by measuring the results when specifications depart from a well-understood baseline model. The following sections specify the chosen baseline model as well as the construction of our departures from that model.

* 1. **Baseline Model**

A baseline model was constructed using the NetworkX package in Python. The base model topology is a 32 by 32 regular lattice with edges connecting von Neumann neighborhoods and a closed boundary condition. As in Xie & Zhou [1], 15% of nodes are reserved as excess housing. The remaining 85% of nodes are randomly assigned either a red or blue occupant. As in Schelling [9], each occupant is assigned an identical tolerance threshold, , indicating a tolerance for at most ~39.62% opposite colored neighbors. To draw a more direct comparison, this -value is obtained by taking the mean tolerance threshold from the populations of agents generated using Xie & Zhou [1] tolerances as described in the next section.

At time , the current neighborhood proportion of dissimilar neighbors for the th occupant in the th neighborhood in , , is given as

where is the number of dissimilar neighbors and is the total number of neighbors in the th neighborhood in . At each time step , is calculated for all occupants and the set of candidate occupants at time , , is constructed. A candidate occupant is randomly selected. For all vacant nodes at time , , neighborhood composition is calculated to create a list of candidate vacancies, , from which a destination node, is randomly selected. The candidate occupant at the th node moves to the selected vacancy and leaves a vacancy in its place. If at any time , , all nodes are satisfied, and no additional trades will be found. If at any time , , the occupant at the selected node,, is unable to locate a satisfactory destination and remains in place.

* 1. **Heterogeneous Tolerances**

Following Xie & Zhou [1], agents are provided with heterogeneous tolerances (tolerance thresholds) aligned with the Guttman scale and rank-ordered logit model derived from Bruch & Mare’s [2] Detroit data. The cumulative distribution function for this model is shown in Figure 2. Tolerance thresholds were assigned by drawing values from a uniform distribution over a given interval: For 10.47% of individuals, fell within [0.0,0.07); for 18.10% of individuals, fell within [0.07,0.21); for 26.73% of individuals, fell within [0.21, 0.36); for 13.86% of individualsfell within [0.36,0.57); for 26.59% of individuals, fell within [0.057,1.00]. For these individuals, the simulation procedure described in the previous section was implemented with replacing . For the 4.25% of individuals in the Detroit data found not to conform to the Guttman scale, no static tolerance threshold was set. Instead, Xie & Zhou’s rank-ordered logit model (Eq. 4 in [1]) was implemented to determine the probability of transition to each candidate neighborhood. The transition destination, , is then randomly selected with probability, , given by the model:

Here, , represents the normalized probability that the occupant will move to the neighborhood in the set of candidate vacancies. This model effectively weights the probability of a move to each candidate vacancy by its proximity to an estimated central tolerance threshold, , on the cusp of the third and fourth intervals noted above.

* 1. **Heterogeneous Topology**

Primarily to make topological differences more explicit, we depart from Gandica, Gargiulo, & Carletti [7], who implemented a metapopulation model, and instead select random neighborhoods to densify. To do so, for each randomly selected node, a set of immediate neighbors, , is constructed. For each neighbor, , a set of second-order neighbors, is constructed. The union of these sets represents a cluster of nodes, . Finally, a set of edges, is constructed. These edges are then added to A single iteration of this procedure is considered a single densification as shown in Figure 3.

This method ensures that neighborhood densities are distributed consistently across the network, so nodes and their neighbors cannot have unrealistic differences in degree. The result of multiple densifications is substantially greater variation in node degree across the network as well as a marked increase in the mean node degree. The result is a variety of neighborhood sizes. While in the base model, each neighborhood is a von Neumann neighborhood bordering another von Neumann neighborhood (except at the boundary), randomly densified lattices have a variety of neighborhood boundary relationships, e.g., a von Neumann neighborhood might be adjacent to a Moore neighborhood. This enables a richer diversity of neighbor relationships.

1. **Experiments**

Ten unique simulation parameterizations were used, half using uniform Schelling [9] tolerance threshold assignments and half using heterogenous Xie & Zhou [1] tolerance threshold assignments. For each of these groups, five separate batches of 100 simulations were conducted with increasing levels of densifications, each simulation observed 4000 timesteps.

* 1. **Heterogeneous Tolerances**

When heterogeneous tolerances were represented, they were always drawn from the same cumulative distribution function as described in Chapter 3. In the homogeneous Schelling case . The adoption of Xie & Zhou’s tolerance model increases to at least . This value is estimated by discretizing assigned tolerance thresholds into 37 identically sized bins and allowing non-Guttman scale agents to occupy an additional distinct bin. The frequencies of Guttman-scale assigned tolerance thresholds are shown in Figure 4.

* 1. **Heterogeneous Topologies**

Each batch had increasing numbers of densifications: 0, 32, 64, 96, and 128. As the number of densifications increased, substantial differences in degree distributions were observed, as shown in Figure 5 and Table 2. Figure 6(a-c) shows relative frequencies of neighborhood-size pair configurations at 0, 32, and 128 densifications and can be said to exhibit degree assortativity [23]. To ensure adequate heterogeneity is produced by the densification procedure, Table 3 displays the associated Shannon entropies for the distributions of neighborhood sizes and neighborhood-size pair configurations at each level of densification. It is clear that the densification procedure described in Chapter 3 effectively produces the desired heterogeneity of topology. Further, it is evident in that the primary explosion of heterogeneity occurs at 32 densifications and beyond this level, returns of heterogeneity diminish. The level of topological heterogeneity produced at 128 densifications exceeds the level produced by any network generator in [7], see Section 4.2.1.

For the range of parameterizations used in our simulations, it is evident that successive increases in densifications lead to progressively smaller increases in heterogeneity of neighborhood size and neighborhood-size pair configurations. This is consistent with the “cannibalization” noted previously – the creation of denser areas consumes sparser ones.

* + 1. **Reconstructing the Metapopulation Model**

It is important to note some restrictions to heterogeneity associated with the metapopulation model in [7]. First, total housing stock is given by where is the total number of nodes and is the node-level population limit. The proportion of vacant housing stock is fixed, as is the total population with the vacancy rate being 90%. Given this, it is obvious that, at most, 10% of nodes could be completely filled through perfect organization. This imposes a limit on total heterogeneity. Further, individuals populate nodes randomly, so by reconstructing this initialization procedure, we find that per-node population is characterized by an approximately normal distribution with and . Thus, though many neighborhood sizes are *possible*, few are *likely* at initialization. Further, because the total population level is fixed, as the population of less-likely (larger) nodes increases, they cannibalize *heterogeneity* at the opposite end of the range, i.e., one node with 100 individuals costs 10 nodes with 10 individuals.

To examine the approximate levels of heterogeneity obtained by the authors’ initialization procedure, grid, small world, random , and scale-free networks were reconstructed. This reconstruction yielded observable differences in initial levels of heterogeneity via Shannon’s entropy measure, see Table 4. Notice that the variance of the distribution of neighborhood sizes is not consistent with its Shannon entropy, this highlights further highlights the distinction between variance and entropy.

Notice that this reconstruction produced mismatched values of and . Differing wiring procedures produced different levels of per increase in . If we selected as the measure of topological heterogeneity, these results are consistent with the differences in asymptotic averaged node magnetization shown in [7], see Figure 1. As heterogeneity of neighborhood-size pairs increases, a larger reduction in , up to a threshold, is required to obtain an equivalent level of node magnetization. This is an indication that heterogeneity of neighborhood-size pairs rather than heterogeneity of neighborhood sizes may be a more important factor in mixing dynamics.

* 1. **Observables**
     1. **Connected Components**

At each time step the number of connected components was recorded. To do so, as in section 4.2.1, the subgraph created when all vacancies are removed is analyzed. When vacancies become organized in a manner that enables, by their removal, the partitioning of the graph, additional connected components may be created. The number and size of connected components were observed throughout each simulation run as a measure of emergent fragmentation due to the spatial organization of vacancies.

* + 1. **Assortativity**

For convenience, Newman’s [23] assortativity coefficient, , is used as a measure of segregation levels during the simulations. is given by Eq. 2 in [23]:

where denotes the fraction of edges which connect a node of type to one of type , , and . Thus, for a graph with occupants with binary attributes, and . When node color populations are approximately equal, we may make the following approximation with :

Values of close to 1 indicate high levels of segregation while values of close to 0 indicate approximately random mixing. To account for vacant nodes on the network, the assortativity coefficient for the subgraph containing only occupied nodes is used. For all simulations where the derived subgraph consisted of multiple connected components, was calculated for each connected component and a weighted average was constructed:

where is the total number of occupied nodes and and are the assortativity coefficient and size of the th component, respectively. For connected components with homogenous composition, a value of 1 was assigned for . To observe the process of segregation, was recorded at each time step.

* + 1. **Organization**

Shannon entropy, , of the distribution of the tolerance levels for each connected pair was observed for the population of graphs at random both at initialization and at the 4000th timestep. Since tolerances are drawn from a continuous distribution, tolerance levels are discretized within 25 equal-sized partitions to calculate the Shannon entropy. Vacancies and non-Guttman tolerance individuals are each assigned to monolithic bins. More explicitly,

where is the pair of discretized tolerances, , observed.

As another indicator of organization, final mean degree was observed for various node types for comparison with the final mean graph degree. Mean degree was observed for vacancies, occupied nodes, nodes occupied by highly tolerant individuals (), and nodes occupied by highly intolerant individuals ().

1. **Results**
   1. **Segregation (Assortativity)**

As anticipated, the observed mean final assortativity was higher for all Schelling-tolerance simulation runs. As in Xie & Zhou [1], heterogeneity of tolerances did, on their own, result in reduced assortativity. Consistent with Gandica, Gargiulo, & Carletti’s [7] observations, increasing topological heterogeneity alone did not produce results with reduced assortativity when , in fact, any increase in topological heterogeneity for Schelling-tolerance simulations resulted in a marked increase in realized assortativity. Finally, similar to the results in Sayama and Yamanoi [16], the combination of both dimensions of heterogeneity resulted in progressive reductions in assortativity. Xie & Zhou-tolerance simulations generated assortativity more slowly and resulted in reduced levels of segregation compared with Schelling-tolerance simulations at every level of densification. The lowest levels of assortativity observed occurred with Xie & Zhou [1] tolerances and 128 densifications. Results are illustrated in Figures 7-8.

The introduction of heterogeneity of topology had the effect of increasing assortativity in all Schelling-tolerance simulation runs due to the presence of densified clusters that increased the number of edges connecting nodes with similar occupants. These increases in assortativity began to disappear when heterogeneities were combined, especially when significant topological heterogeneity was present.

* 1. **Fragmentation**

For certain levels of densification ( greater average fragmentation was observed when Xie & Zhou [1] tolerances were used. For others () greater average fragmentation was observed when Schelling [9] tolerances were used. These differences do not appear to be a result of self-organized partitioning. As expected, large numbers of connected components were not observed as graph connectivity increased. These results are illustrated in Figure 9.

* 1. **Organization**
     1. **Hub Composition**

The mean degree for each graph and its vacancies was calculated, see Figure 10. The mean vacancy degree showed greater variability than the graph degree for each group. No significant difference was observed between the accumulations of vacancies in Xie & Zhou tolerance simulations and Schelling tolerance simulations. When substantial levels of densification were present, the mean vacancy degree could exceed the mean graph degree. Otherwise, the mean vacancy degree tended lower. For simulations with Xie & Zhou tolerances on densified graphs, on average, individuals in the group with the highest tolerance thresholds () had distinctly higher degree centralities than those in the group with the lowest tolerance thresholds (). This difference became more pronounced as the number of densifications increased. These results are illustrated in Figures 11-12.

* + 1. **Paired Tolerances**

In addition to the topological organization of tolerances, the pairwise organization of tolerances can also be observed. At every level of densification, the distribution of pairs’ tolerances became organized as segregation increased. This effect became less pronounced as the number of densifications increased. Additionally, the organization of tolerances can be observed visually in Figure 12. At all levels of densification, distinct regions of like-tolerance nodes can be observed. This organization is further illustrated in Figure 13 where the relative frequencies of tolerance-pairs for dissimilar neighbors are shown. The overall level of organization exhibited at each level, the difference in levels of heterogeneity, is shown in Table 5. As densifications increase, final graphs are less restrictively organized.

1. **Discussion**

Two key dynamics, either enabled or augmented by the combination of behavioral and topological heterogeneities, produce substantial, novel reductions in segregation on networks: migration is ordered, and tolerance repels intolerance.

* 1. **Ordered Migration**

The probability that an individual will exceed its tolerance threshold, can be given as a function of the probability, , that at least neighbors will be dissimilar:

where is the individual’s tolerance threshold and is the neighborhood size. Since our models begin with a randomly mixed population where and increases monotonically, we need only consider cases where , as shown in Figure 14.For , with only heterogeneity of topology, individuals in smaller neighborhoods will generally become more likely to exceed their tolerance thresholds than those in larger neighborhoods as assortativity increases over time. In this way, an additional dynamic is included: the out-migration of non-dominant-type individuals in densified clusters will precede in-migration to those clusters by dominant-type individuals. This behavior enables the increase in assortativity observed in Schelling-tolerance simulations with topological heterogeneity.

When both topological and behavioral heterogeneities are represented, dynamics are richer. The probability of an individual’s residence in a location that exceeds its tolerance threshold can be roughly ordered from greatest to least: low tolerance, large neighborhood; low tolerance, small neighborhood; moderate tolerance, small neighborhood; moderate tolerance, large neighborhood; high tolerance, small neighborhood; and high tolerance, large neighborhood.

* 1. **Tolerance Repels Intolerance**

When heterogeneity of tolerances is represented, pairs of dissimilar individuals with high tolerance thresholds create another interesting dynamic: these nodes repel intolerant nodes and attract tolerant neighbors. This dynamic is restrained on a 2D grid with von Neumann neighborhoods. Such pairs can have no common neighbors, but some of the neighbors of each will share an edge, see Figure 15. With network topologies where dissimilar adjacent nodes may share a neighbor, that neighbor must possess sufficient tolerances to accommodate at least one pair of dissimilar neighbors.

When both heterogeneous tolerances and heterogeneous topology are represented, the influence of tolerant nodes is amplified by densifications. A densified cluster populated by dissimilar, tolerant nodes acts as a strong repellant for intolerant nodes as tolerant pairs will directly share more neighbors. Second, a densified cluster occupied by similar nodes may continue to attract similar nodes but can no longer repel a dissimilar individual with a sufficiently large tolerance threshold. The introduction of relatively few highly tolerant dissimilar individuals to such a cluster will motivate catastrophic outmigration of intolerant individuals. These dynamics lead some densified clusters to become stores of tolerance and diversity. As the level of densification increases, the presence of tolerant nodes and densified clusters makes extremely homogeneous areas rarer. As this occurs, the most intolerant nodes can no longer find suitably homogenous candidate vacancies and become increasingly connected to tolerant, dissimilar nodes, see Figure 13. Thus, both the organization of tolerances and the organization of color-types in the final networks are reduced, see Table 5.

* 1. **Conclusion**

Network models of segregation which combine heterogeneity of tolerances with heterogeneity of topologies show substantially different behavior than those which employ only one dimension of heterogeneity. Ordered migration and the resultant clusters of tolerance break down intolerant homogeny. This has bearing on the important question raised previously: to what extent can the collective dynamics of individual preferences lead to residential segregation? These results indicate that these collective dynamics may not contribute to residential segregation as much as previously thought if sufficient heterogeneity is represented. At a minimum, these results indicate the necessity of representing both heterogeneities of tolerances and heterogeneity of topologies in network models of residential segregation, as the omission of one or the other will result in the loss of these important model behaviors.

It is notable that the dynamics which break down segregation along racial lines appear to create segregation in another way: the most tolerant individuals can become separated from the least tolerant individuals. This finding, paired with the results in Sayama & Yamanoi [16], may explain ongoing cultural fragmentation between urban and rural areas. While a diversity of tolerances is necessary for simultaneous maintenance of cultural diversity and social cohesion, we observe segregation within the diversity of tolerances themselves. This segregation ensures that opportunities for cultural diffusion between the intolerant subsets of population will be scarce. This scarcity could play an important role in the maintenance of cultural diversity.

**Acknowledgments**

Paste your acknowledgments here.

**References**

**References**

1. Xie, Y., & Zhou, X. (2012). Modeling individual-level heterogeneity in racial residential segregation. Proceedings of the National Academy of Sciences - PNAS, 109[29], 11646-11651.
2. Bruch, E. E., & Mare, R. D. (2006). Neighborhood choice and neighborhood change. The American journal of sociology, 112[3], 667-709.
3. Clark, W. A. (1991). Residential Preferences and Neighborhood Racial Segregation: A Test of the Schelling Segregation Model. Demography, 28[1], 1-19.
4. Clark, W. A., & Fossett, M. (2008). Understanding the social context of the Schelling segregation model. Proceedings of the National Academy of Sciences - PNAS, 105[11], 4109-4114.
5. Fossett, M. (2006). Ethnic Preferences, Social Distance Dynamics, and Residential Segregation: Theoretical Explorations Using Simulation Analysis. The Journal of Mathematical Sociology, 30[3-4], 185-273.
6. Gargiulo, Gandica, Y., & Carletti, T. (2015). Urban skylines from Schelling model.
7. Gandica, Y., Gargiulo, F., & Carletti, T. (2016). Can topology reshape segregation patterns? Chaos, Solitons, and Fractals, 90, 46-54.
8. Sakoda. (1971). The checkerboard model of social interaction. The Journal of Mathematical Sociology, 1(1), 119–132.
9. Schelling, T. (1971). Dynamic models of segregation. The Journal of mathematical sociology, 1[2], 143-186.
10. Zhang, J. (2004). A Dynamic Model of Segregation. The Journal of Mathematical Sociology, 28[3], 147-170.
11. Zhang, J. (2004). Residential segregation in an all-integrationist world. Journal of Economic Behavior & Organization, 54[4], 533-550.
12. Ashby, W. R. (1958). Requisite Variety and its implications for the control of complex systems. Cybernetica, 1(2), 83-99.
13. Conant, R. C., & Ashby, W. R. (1970). Every good regulator of a system must be a model of that system. International Journal of Systems Science, 1(2), 89-97.
14. Hartley, R. V. L. (1928). Transmission of Information. Bell System Technical Journal, 7(3), 535-563.
15. Shannon, C. E. (1948). A Mathematical Theory of Communication. The Bell System Technical Journal, 27(3), 379-423.
16. Sayama, H., & Yamanoi, J. (2019). Beyond Social Fragmentation: Coexistence of Cultural Diversity and Structural Connectivity Is Possible with Social Constituent Diversity.
17. Yamanoi, J., Sayama, H. (2013). Post-merger cultural integration from a social network perspective: a computational modeling approach. Comput Math Organ Theory 19, 516–537.
18. Hegselmann. (2017). Thomas C. Schelling and James M. Sakoda: The Intellectual, Technical, and Social History of a Model. Journal of Artificial Societies and Social Simulation, 20(3).
19. Frankenberg, E., & Taylor, K. (2018). De Facto Segregation: Tracing a legal basis for contemporary inequality. Journal of Law and Education, 47[2], 189-233.
20. Watts, & Strogatz, S. H. (1998). Collective dynamics of “small-world” networks. Nature (London), 393(6684), 440–442.
21. Erdos, P., and Renyi, A. (1960). On the evolution of random graphs. Publ. Math. Inst. Hung. Acad. Sci 5 :17-61.
22. Barabasi L. and Albert R. (2002). Statistical mechanics of complex networks. Rev. Modern Physics, 7:47-97
23. Newman, M. E. (2003). Mixing patterns in networks. Physical review. E, Statistical, nonlinear, and soft matter physics, 67[2 pt 2], 026126–026126.

**Figures and Tables**

**Diagram

Description automatically generated**

**Figure 1. From Gandica, Gargiulo, & Carletti [7]. Impact of tolerance level and graph architecture on asymptotic averaged node magnetization.**

**Chart, line chart

Description automatically generated**

**Figure 2. Cumulative distribution of Guttman scale tolerance thresholds for *all* simulation instances. Not shown: 4.25% of individuals obeying Xie & Zhou’s rank-ordered logit model transition function.**

**A picture containing colorful, orange, bright

Description automatically generated Chart, radar chart

Description automatically generated Chart, radar chart

Description automatically generated**

**Figure 3. Left: an initial portion of a lattice; center: its densified counterpart; right: the metapopulation representation of the densification obtained by replacing the cluster with a single metapopulation node. Each node within the densified cluster has a link to all other nodes in the cluster. This is equivalent to their replacement with a single node as a container for the cluster.**

**A graph of a number of blue bars

Description automatically generated**

**Figure 4. Histogram of Guttman scale tolerance thresholds initialized for a single simulation instance. Not shown: 4.25% of individuals obeying Xie & Zhou’s rank-ordered logit model transition function.**

**A graph of different colored shapes

Description automatically generated**

**Figure 5. Impact of densifications on graph degree distribution. Distributions for 32, 64, 96, and 128 densifications are bimodal. There is an observable transition between dominant modes as densifications increase.**

**A diagram of a number of squares

Description automatically generated**

**Figure 6(a). Log frequency of node-neighbor degree pairs without densifications.**

**A red and black grid

Description automatically generated**

**Figure 6(b). Log frequency of node-neighbor degree pairs with 32 densifications.**

**Chart

Description automatically generated with medium confidence**

**Figure 6(c). Log frequency of node-neighbor degree pairs with 128 densifications.**

**A graph with numbers and symbols

Description automatically generated**

**Figure 7. Initial rate of change in assortativity, , and final assortativity, , for each simulation group.**

A graph of different colored shapes

Description automatically generated

**Figure 8. Final graph assortativity for each simulation group.**

**A graph of different colored lines

Description automatically generated**

**Figure 9. Final connected components for each simulation group.**

**Chart, box and whisker chart

Description automatically generated**

**Figure 10. Final mean vacancy and node degree for each simulation group.**

**A diagram of different colored shapes

Description automatically generated**

**Figure 11. Final mean degree by node tolerance and level of densification.**

**A collage of multiple images of different colors

Description automatically generated**

**Figure 12. Sample final graph images for 0, 32, 64, 96, and 128 densifications using Xie & Zhou tolerances. Redder colorings represent less tolerant individuals, greener colorings represent more tolerant individuals. Blue coloring represents non-Guttman individuals. Vacancies are removed.**

Chart, treemap chart

Description automatically generatedA red and white squares

Description automatically generated

A graph of a number of red squares

Description automatically generatedGraphical user interface

Description automatically generated with medium confidence

**Figure 13. Heatmaps show the log frequencies of tolerance-level pairs for dissimilar neighbors. Top row: 0 densifications; bottom row: 128 densifications; left column: initial distribution; right column: final distribution.**

A diagram of a number of dissimilar neighbors

Description automatically generated

**Figure 14. Vertical axis: Probability of exceeding tolerance threshold, . Horizonal axis: probability of nth dissimilar neighbor, . Note that as assortativity *increases*, the probability of dissimilar neighbors *decreases.***

A diagram of a complex structure

Description automatically generated

**Figure 15. Left, a pair of highly tolerant dissimilar neighbors (highlighted in green). The upper node in the pair has an intolerant, similar neighbor (highlighted in orange). Given this configuration, for a red node to occupy the space highlighted in purple, it must possess a high tolerance threshold. Right, a group of tolerant dissimilar neighbors inside a densified cluster. The upper node in the pair has a vacant neighbor (highlighted in orange). Given this configuration, an intolerant node will likely never occupy the space highlighted in orange, regardless of its color. The tolerant nodes are highly entrenched.**

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

**Table 2. Mean, standard deviation, and median degree observed for each level of densification. increases steadily as the number of densifications increases; increases abruptly from 0 to 32 densifications, but then increases slow substantially; has abrupt increases at 64 and 128 densifications.**

|  |  |  |
| --- | --- | --- |
| **D** |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

**Table 3. Shannon entropy of node degrees and neighborhood-size pair configurations.**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Graph Type |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

**Table 4. Results of simulated graph reconstruction. For each graph type, 100 graph instances were generated to obtain , , and where is the node degree. In addition, Shannon entropy of the distribution of neighborhood-size pair configurations was calculated. Differences in heterogeneity of neighborhood sizes offer less explanation for the authors’ results than heterogeneity of neighborhood-size pair configurations.**

|  |  |  |
| --- | --- | --- |
| D |  |  |
| 0 | 7.651 | 5.842 |
| 32 | 7.646 | 6.155 |
| 64 | 7.654 | 6.138 |
| 96 | 7.636 | 6.163 |
| 128 | 7.642 | 6.254 |

**Table 5. Initial and final Shannon entropies for the distribution of pairs' tolerances. At each level, organization of tolerances is observed.**